Forced oscillator

24TH AUGUST 2020

Review :
The driven, undamped harmonic oscillator

- The equation of motion for harmonically driven, undamped oscillator may be written as

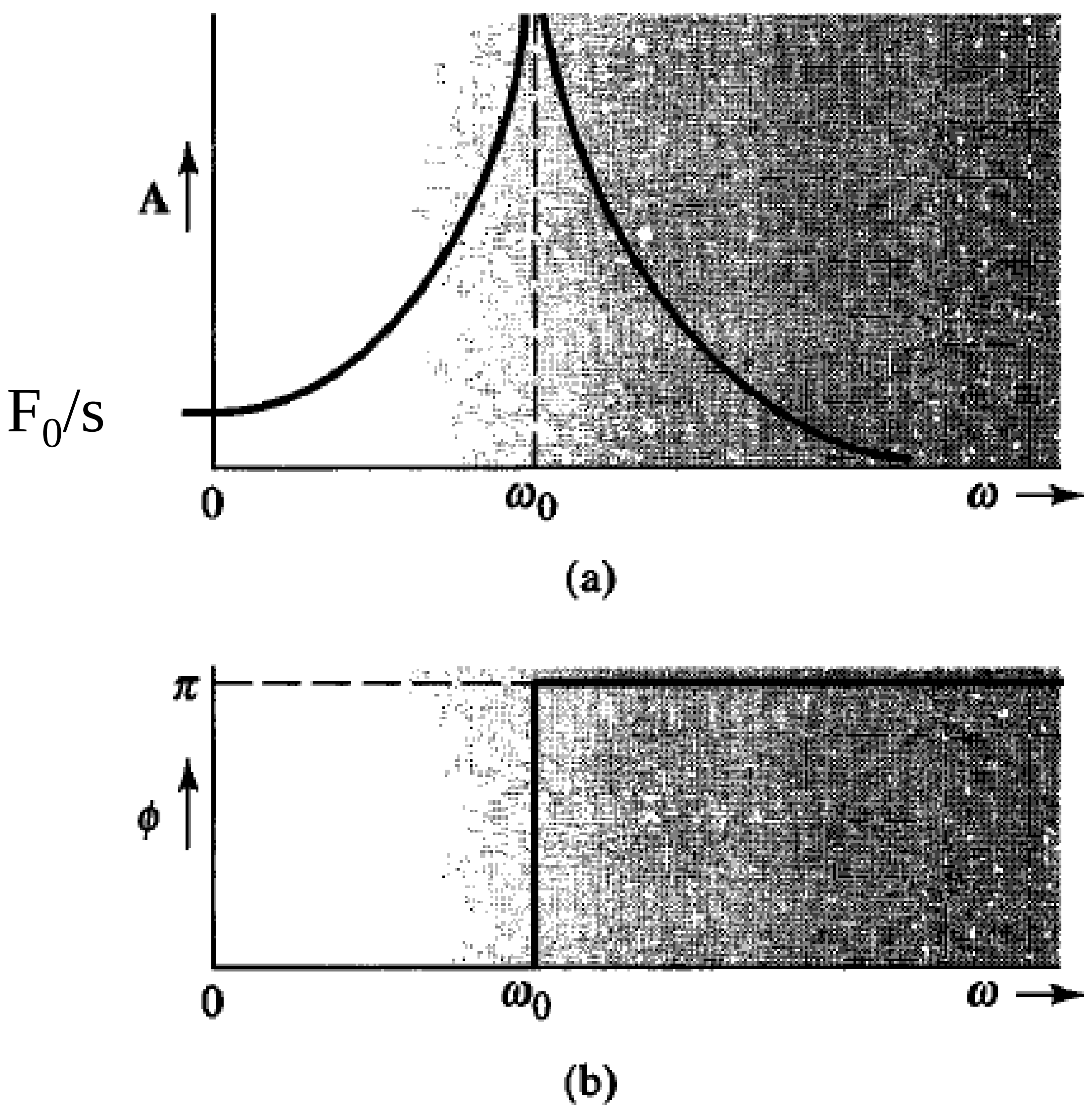
- If a general displacement x is given as $x(t) = A\cos(\omega t + \phi)$, the amplitude A is found to be

$$A = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right)} \text{ or } A = \frac{F_0/m}{\left(\omega^2 - \omega_0^2\right)}$$

depending on $\phi = 0$ or π

and ω_0 = frequency of free oscillation

- The phase shift of "0" indicates that the displacement and the driving force are **in phase**.
- The phase shift of "π" indicated that the displacement and the driving force are **out of phase by π**.

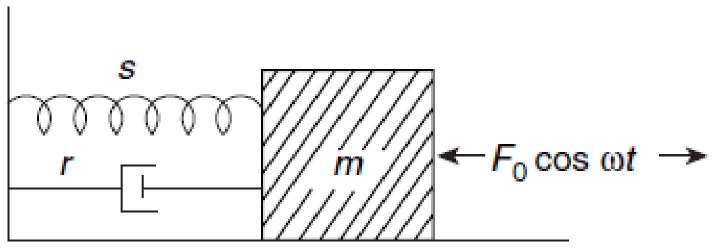


- (a) the amplitude of a driven oscillator versus ω with no damping.
- (b) The phase lag of the displacement relative to the driving force versus ω.

$$A = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right)}\ ,\ \phi = 0 \text{ and } \omega < \omega_0$$

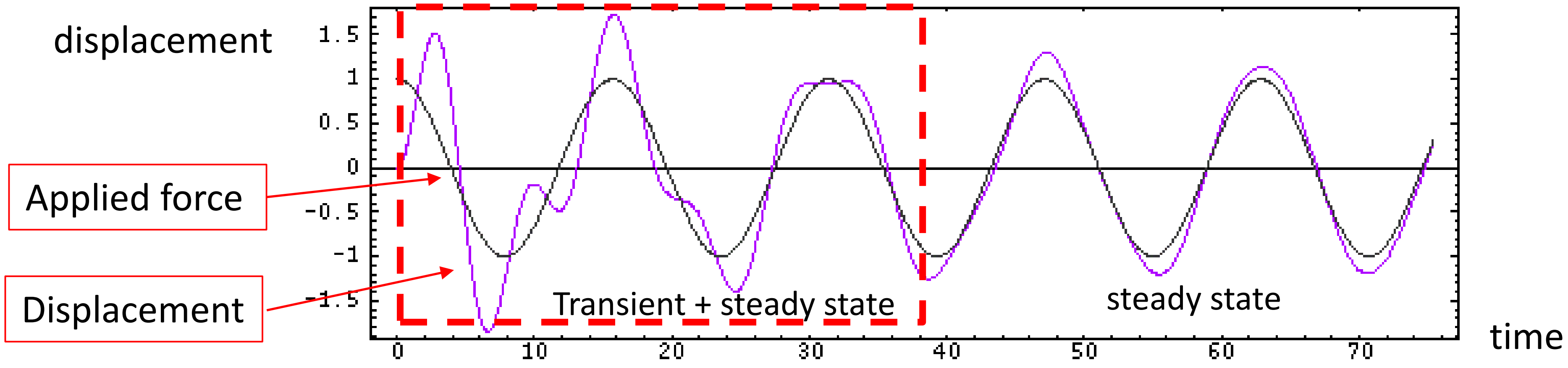
$$A = \frac{F_0/m}{\left(\omega^2 - \omega_0^2\right)},\ \phi = \pi \text{ and } \omega > \omega_0$$

Behavior of the forced oscillator



- Consider a mechanical forced oscillator with force $F_0 \cos\omega t$ applied to **damped oscillator**.
- The equation of motion is given by $m\ddot{x} + r\dot{x} + sx = F_0 \cos \omega t$
- The complete solution for the case is composed of

(1) **Transient term** and (2) **Steady state term**



Driven, damped harmonic oscillator

- Generally, the applied force may be written as **F_0exp(iωt).**
- Therefore, the equation of motion becomes $m\ddot{x} + r\dot{x} + sx = F_0 \exp(i\omega t)$
- Suppose a general solution of the differential equation is $x = A\exp(i\omega t)$
- By substituting the general solution into the equation of motion, we obtain the amplitude *A* as follows

$$A = \frac{F_0}{\left(s - m\omega^2 + i\omega r\right)} = \frac{-iF_0}{\omega\left(r + i\left(\omega m - \frac{s}{\omega}\right)\right)}$$

Description of the steady state term

- Therefore, the displacement can be written as

$$x = A\exp(i\omega t) = \frac{-iF_0 \exp(i\omega t)}{\omega[Z_m]} = \frac{-iF_0 \exp(i\omega t)}{\omega\left[|Z_m|\exp(i\phi)\right]} = \frac{-iF_0 \exp i(\omega t - \phi)}{\omega|Z_m|}$$

$$Z_m = \left[r + i(\omega m - s/\omega)\right] = \text{mechanical impedance}$$

$$\phi = \tan^{-1}\left[(\omega m - s/\omega)/r\right]$$

Mechanical impedance is a measure of how much a structure resists motion when subjected to a harmonic force. It relates forces with velocities acting on a mechanical system.

Complete information of magnitude and phase of the steady state term

2

1

$$x = \frac{-iF_0 e^{i(\omega t - \phi)}}{\omega |Z_m|}$$

3

(1)

(2)

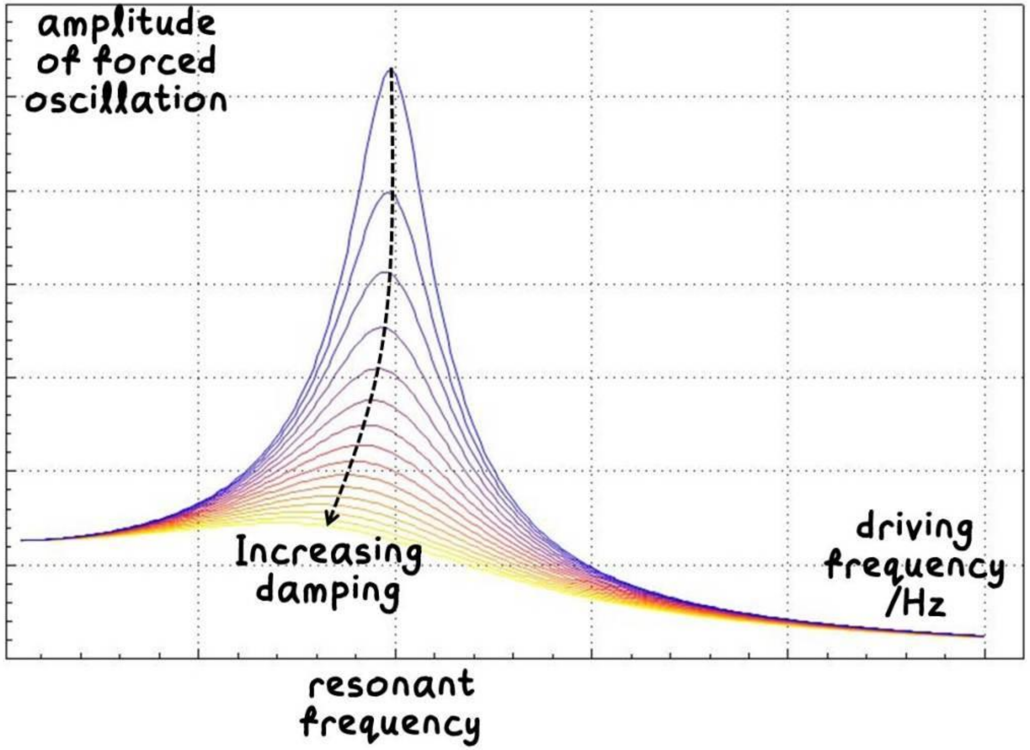
(3)

The amplitude of the *x*

- Recall the amplitude from the steady state,

$$A = \frac{F_0}{\omega |Z_m|} = \frac{F_0}{\omega\left[r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right]^{\frac{1}{2}}} = \frac{\frac{F_0}{m}}{\left[(\omega_0^2 - \omega^2)^2 + \left(\frac{r\omega}{m}\right)^2\right]^{\frac{1}{2}}}$$

- The expression clearly states that the amplitude is a function of the **driving frequency, ω**.
- The amplitude is still finite even though the driving frequency is equal to the frequency of the free oscillation *due to the existence of the r*.



What is the resonant frequency of the forced oscillator?

How to get a resonant frequency of the forced oscillator?

How to represent the forced oscillation resonant frequency in terms of damping frequency?

A note on the applied force

- Instead of using an exponential form, the applied force as written in terms of **cosine or since functions** leads to the steady state solution as follows
- The value of displacement x resulting from $F_0\cos(\omega t)$ is

$$x = \frac{F_0}{\omega\left|Z_m\right|}\sin\left(\omega t - \phi\right)$$

- The value of displacement x resulting from $F_0\sin(\omega t)$ is

$$x = \frac{-F_0}{\omega\left|Z_m\right|}\cos\left(\omega t - \phi\right)$$

- Both solutions satisfy the information given in the previous page.

Mechanical Impedance Z_m

- **Mechanical impedance is a measure of how much a structure resists motion when subjected to a harmonic force. It relates forces with velocities acting on a mechanical system.**
- Mechanical impedance is a complex quantity given by

$$Z_m = \left[r + i\left(\omega m - s/\omega \right) \right]$$

- The real part, the **mechanical resistance**, is independent of frequency.

  The dissipative forces ($r\dot{x}$) are proportional to velocity.
- The imaginary part, the **mechanical reactance**, varies with frequency, becoming zero when equal to the frequency of shm.

https://en.wikipedia.org/wiki/Mechanical_impedance

Relation of the velocity and force

Provided that the driving force is given as $F_0\cos(\omega t)$, the velocity becomes

1) In case of $\phi = 0$, velocity and force are in phase.
2) The amplitude of the velocity is $F_0/|Z_m|$, this leads to the definition of the **mechanical impedance** $Z_m = F/V$

Problem 1

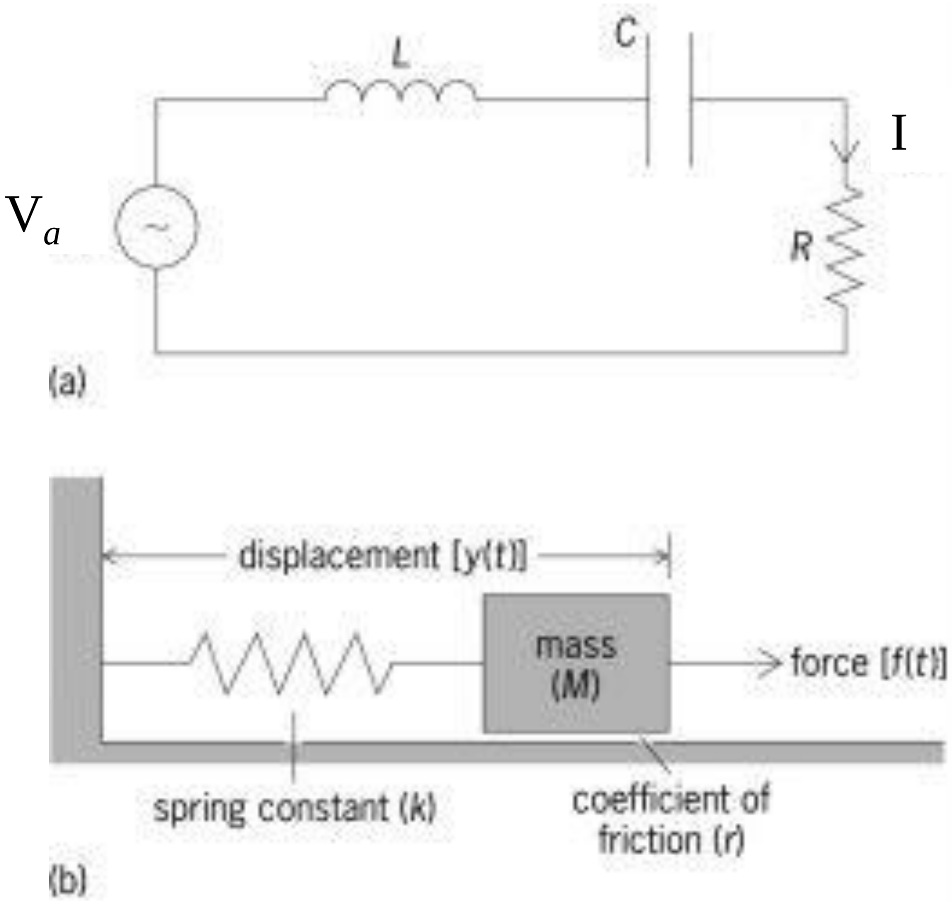
The equation $m\ddot{x} + sx = F_0 \sin(\omega t)$ describes the motion of an undamped simple harmonic oscillator driven by a force of frequency ω.

-Determine the steady state solution and sketch the behavior of the steady state amplitude versus ω.

-Also find the general solution.

Detailed solution of problem 1

A response of RLC series circuit



- The input voltage is equal the sum of the voltage across the inductor, the voltage across the capacitor and the voltage across the resistor.

$$V_L + V_R + V_C = V_a$$

$$L\ddot{q} + R\dot{q} + q/C = V_a$$

- If $V_a = V_0 \exp(i\omega t)$, the solution of the above differential equation is given as

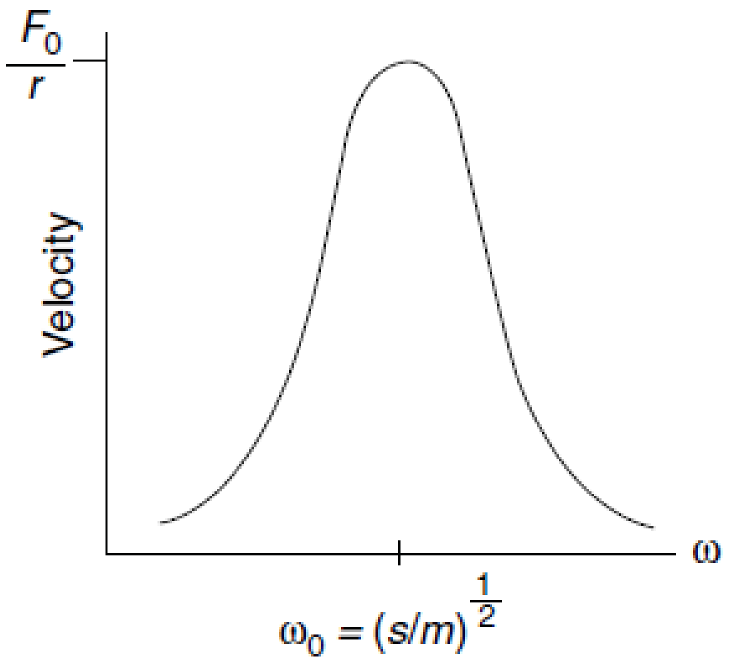
$$q =$$

- Where the electrical impedance Z_e is written as

$$Z_e =$$

To find the solution, we simply compare the electrical system to the mechanical system and substituting m for L, r for R and s for 1/C.

Behavior of velocity v in magnitude versus driving for frequency ω



Velocity of forced oscillator versus driving frequency ω.

- The magnitude of the velocity amplitude varies with frequency ω because $|Z_m|$ is frequency dependent.

$$\frac{F_0}{|Z_m|} = \frac{F_0}{\left[r^2 + (\omega m - s/\omega)^2\right]^{1/2}}$$

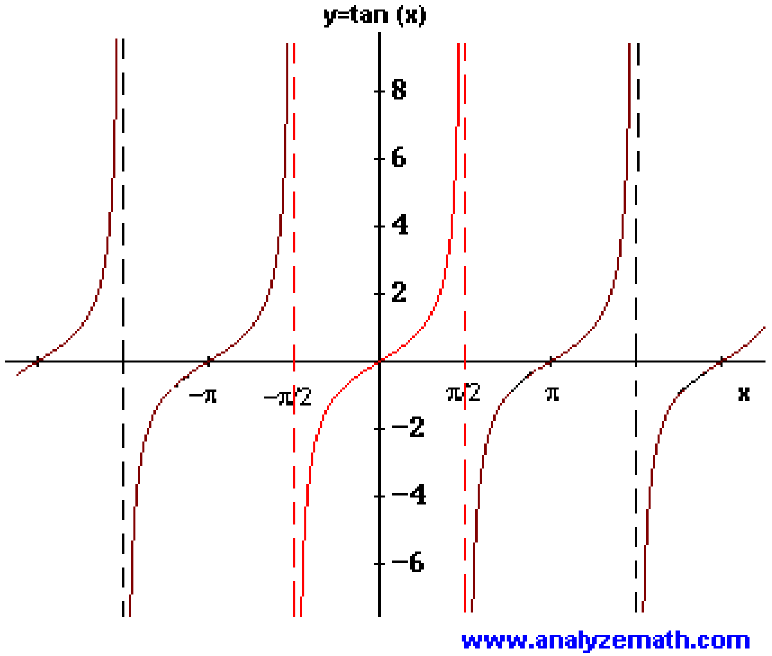
- The impedance is **stiffness controlled** : at low frequency, s/ω dominates.
- The impedance is **mass controlled**: at high frequency, ωm dominates.
- Let me remind you that the driving force : $F_0 \cos \omega t$,

The velocity $(F_0/|Z_m|)\cos(\omega t - \phi)$ where $\phi = \tan^{-1}\left(\frac{\omega m - s/\omega}{r}\right)$,

At resonance, the velocity is in phase with the driving force.

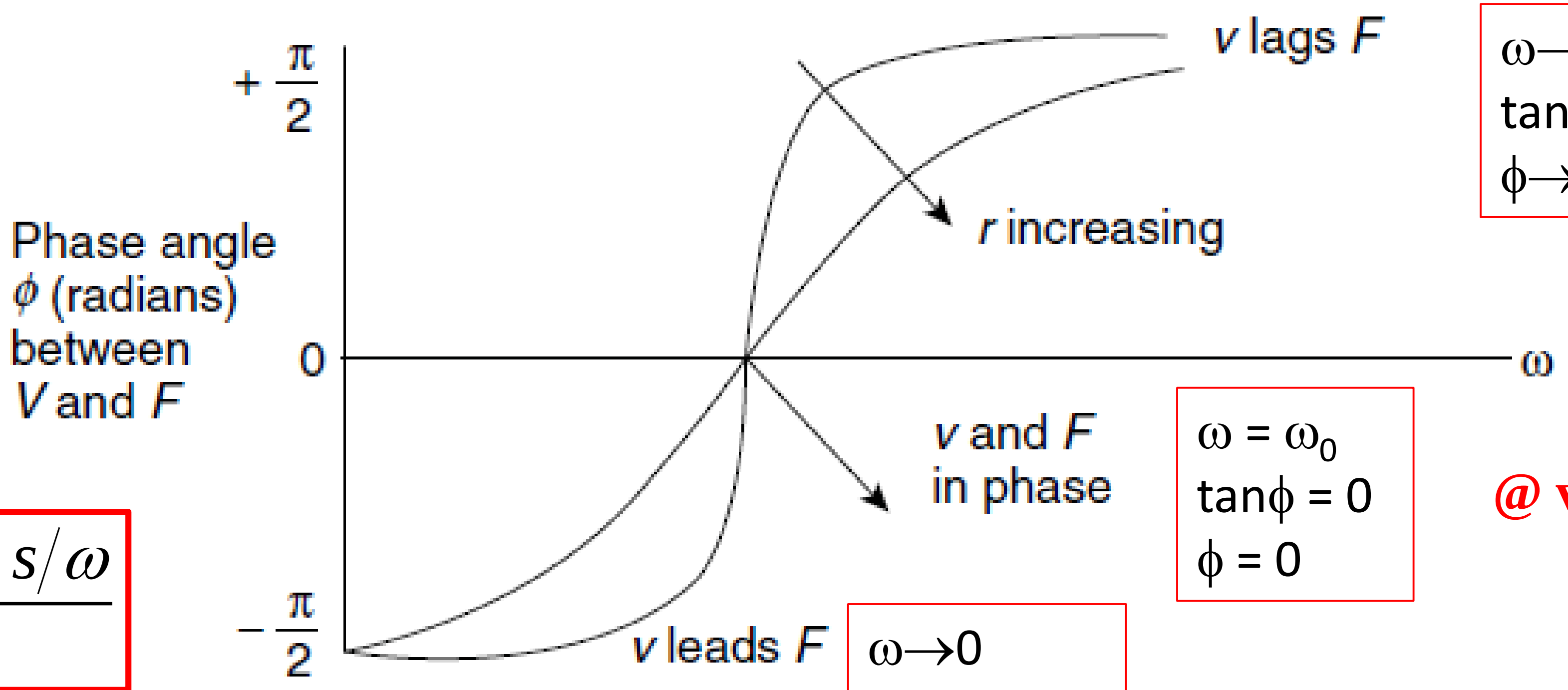
Phase behavior of velocity v versus driving force frequency ω

- According to the relationship between the velocity *v* and force F, $v = \frac{F_0}{|Z_m|}\cos(\omega t - \phi)$
- The applied force is $F = F_0 \cos \omega t$
- Generally, *v* lags F by ϕ and $\tan\phi = \frac{\omega m - s/\omega}{r}$
- Consider 3 situations; ϕ > 0, ϕ < 0 and ϕ = 0.



http://www.analyzemath.com/trigonometry/properties.html

Variation of phase angle ϕ versus driving force frequency ω



$\omega \to \infty$
$\tan\phi \to \infty$
$\phi \to \pi/2$

Mass controlled

$\omega = \omega_0$
$\tan\phi = 0$
$\phi = 0$

@ velocity resonance

$\omega \to 0$
$\tan\phi \to -\infty$
$\phi \to -\pi/2$

Stiffness controlled

$$\tan\phi = \frac{\omega m - s/\omega}{r}$$

At low frequency the velocity **leads** the force (ϕ negative) and at high frequency the velocity **lags** the force (ϕ positive).

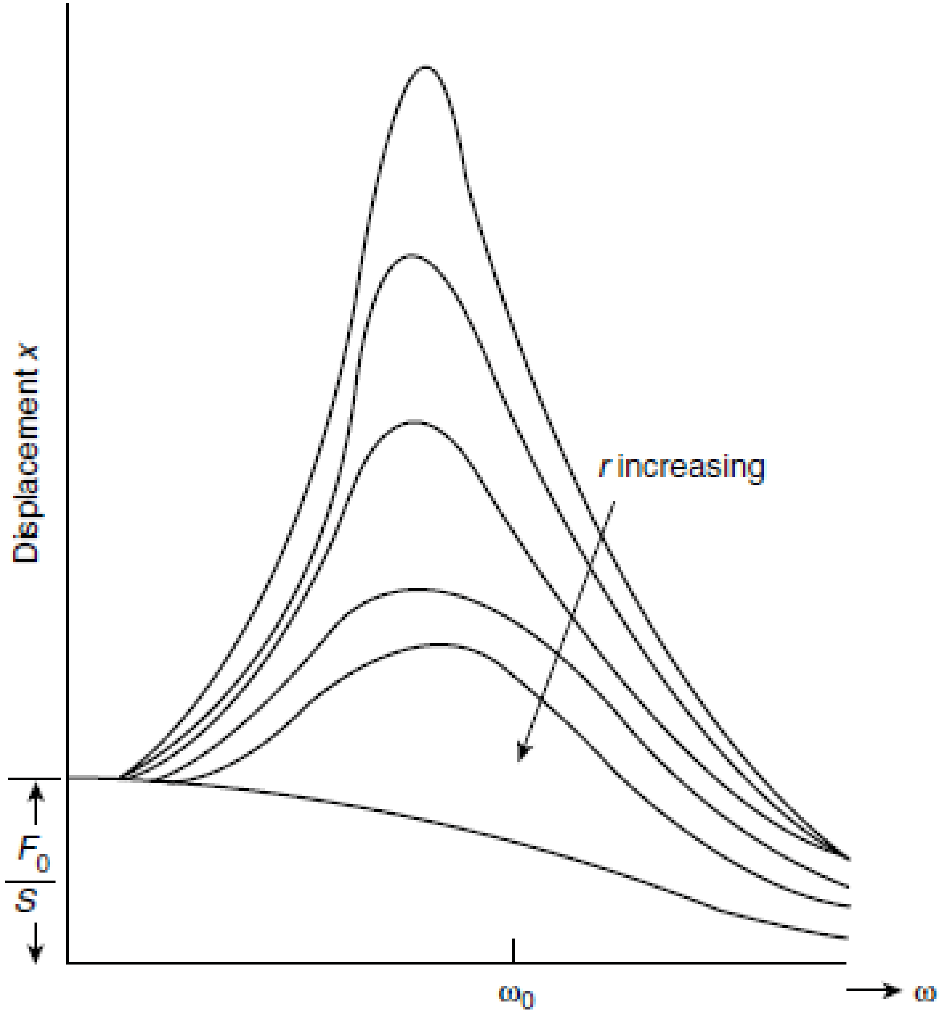
Behavior of displacement in magnitude versus driving force frequency ω

- Recall the displacement $$x = \frac{F_0}{\omega |Z_m|} \sin(\omega t - \phi)$$ **when the driving force is $F_0\cos(\omega t)$,**
- Clearly, the amplitude is given as $\frac{F_0}{\omega |Z_m|}$ and $$|Z_m| = \left[r^2 + (\omega m - s/\omega)^2 \right]^{\frac{1}{2}}$$
- The amplitude function suggests that the graph of x vs ω depends on 3 different ranges of ω.
- What would the amplitude be when ω → 0?
- What would the amplitude be when ω → ∞ ?
- What is the driving frequency at the amplitude resonance?

The amplitude resonance of the displacement

- The displacement resonance occurs when the denominator ωZ_m is a minimum.
- This takes place when $$\frac{d(\omega Z_m)}{d\omega} = \frac{d}{d\omega}\,\omega\left[r^2 + (\omega m - s/\omega)^2\right]^{\frac{1}{2}} = 0$$
- The condition gives the driving frequency ω which gives the **displacement resonance**.
- Therefore, $$\omega^2 = \omega_r^2 = \omega_0^2 - \frac{r^2}{2m^2}$$
- Thus, the displacement resonance occurs at a frequency **slightly less than ω_0, the** frequency of velocity resonance.
- **Express the driving resonance frequency in terms of damping frequency?** **Already done!**

Variation of the displacement of a forced oscillator vs driving force frequency ω



- The maximum displacement at resonance amplitude is given as $$x_{\max} = \frac{F_0}{\omega_r |Z_m|}$$
- Due to $\omega_r|Z_m| = \omega' r$ (**Prove this!**)
- Therefore, $$x_{\max} = \frac{F_0}{\omega' r}$$
  Damping frequency (ω′)
- The amplitude at resonance is kept low by increasing *r*.

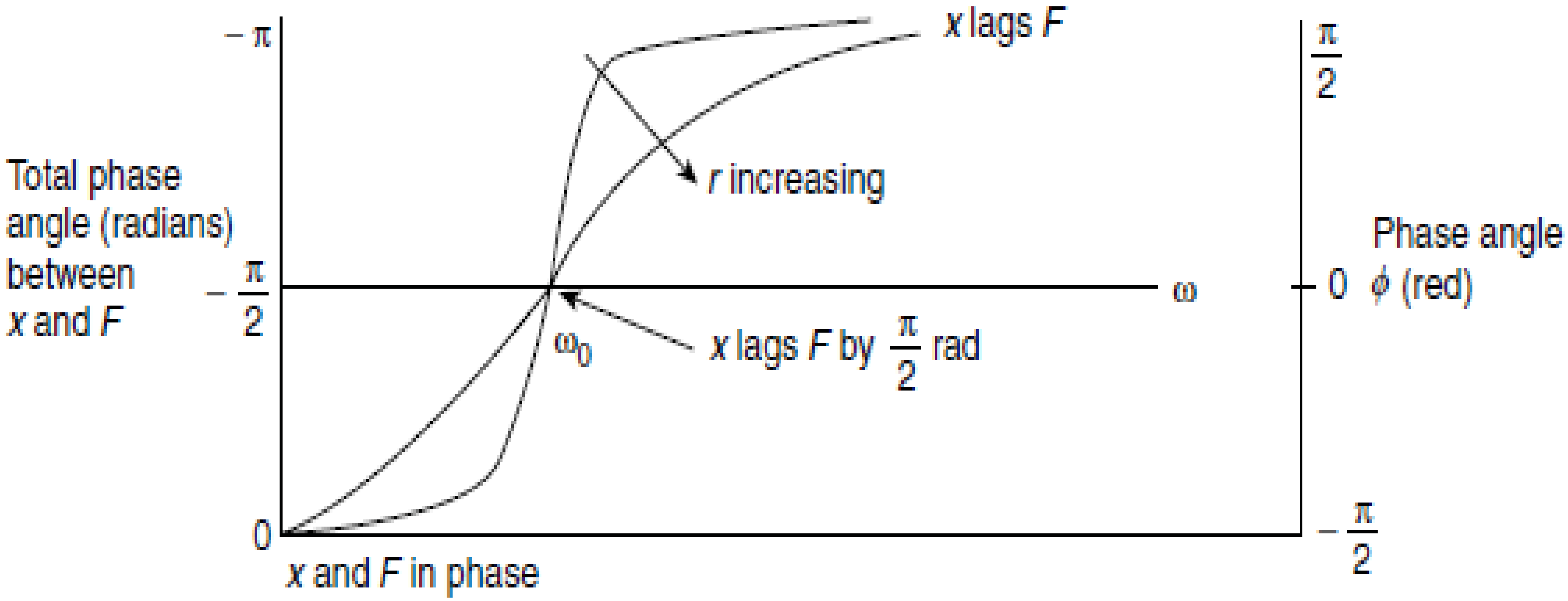
Prove $\omega_r Z_m = \omega' r$

Phase behavior of displacement versus driving force frequency ω

- Recall the displacement $x = \frac{F_0}{\omega|Z_m|}\sin(\omega t - \phi)$ and the driving force $F_0 \cos \omega t$
- Since the displacement *x* lags velocity v by ……   ……

- Consider when ω → 0, the above condition suggests that x lags/leads/is in phase with F.
- Consider when ω → ∞, x lags/leads/is in phase with F.
- Consider when $\omega = \omega_0$, x lags/leads/is in phase with F.

Variation of total phase angle between displacement and driving force vs driving frequency ω



This can be easily explained when considered each condition firstly with the phase difference between v and F and then x and F provided that x and v is always out of phase by $\pi/2$.

Significance of the two components of the displacement curve (1)

- From the displacement $x = \frac{F_0}{\omega|Z_m|}\sin(\omega t - \phi)$

- This expression may be rewritten as $x = \frac{F_0}{\omega|Z_m|}\sin(\omega t - \phi) = \frac{F_0}{\omega|Z_m|}[\sin\omega t\cos\phi - \cos\omega t\sin\phi]$

- Due to $|Z_m| = \left[r^2 + X_m^2\right]^{\frac{1}{2}}; X_m = (\omega m - s/\omega)$ and $\sin\phi = \frac{X_m}{Z_m}; \cos\phi = \frac{r}{Z_m}$

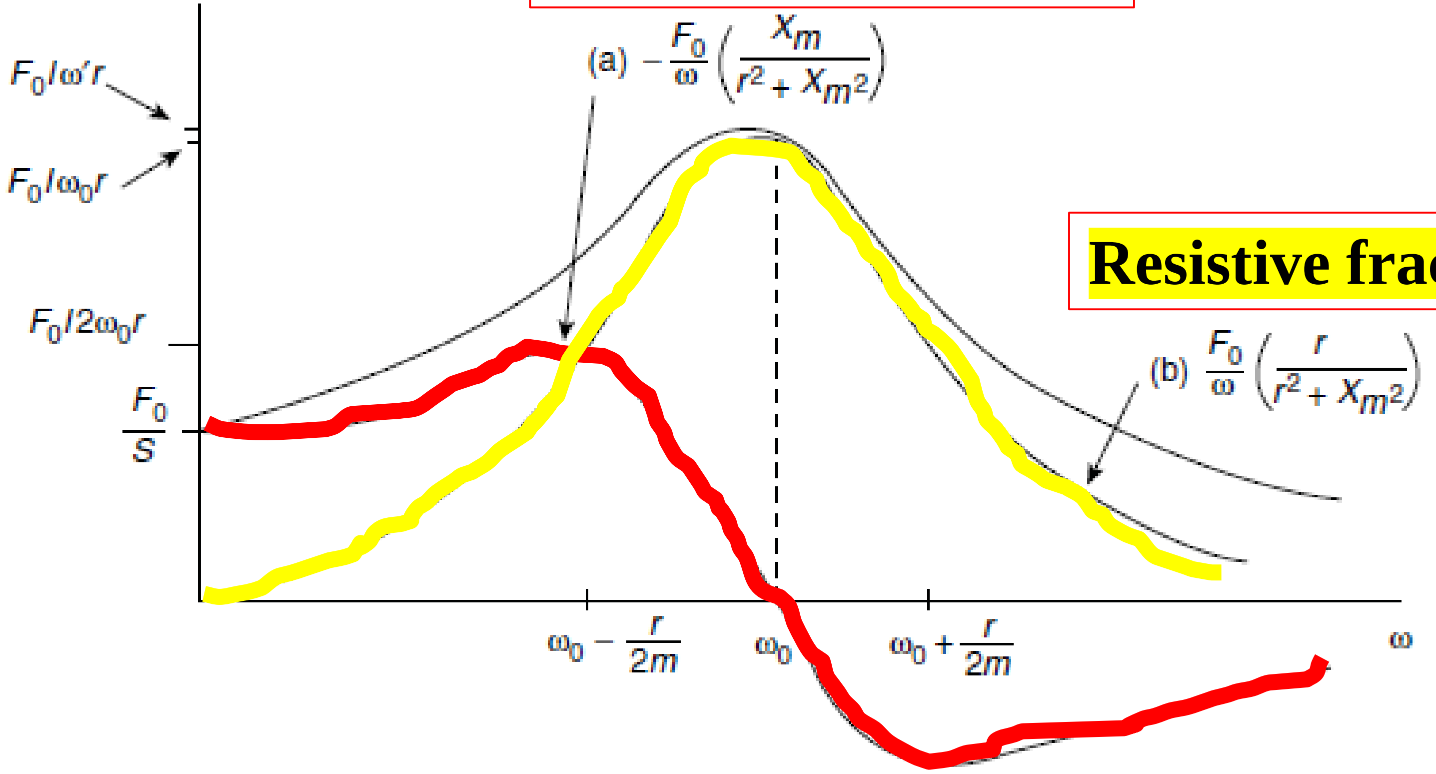
- The displacement is then composed of two terms: **resistive fraction** and **reactive fraction**,

$$x = \frac{F_0}{\omega}\frac{r}{|Z_m|^2}\sin\omega t - \frac{F_0}{\omega}\frac{X_m}{|Z_m|^2}\cos\omega t$$

Resistive term **Reactive term**

Significance of the two components of the displacement curve (2)

Reactive fraction



Resistive fraction

- The amplitude of the **reactive fraction** may be written as

$$-\frac{F_0}{\omega}\frac{X_m}{|Z_m|^2}=\frac{F_0 m\left(\omega_0^2-\omega^2\right)}{m^2\left(\omega_0^2-\omega^2\right)^2+\omega^2 r^2}$$

- The amplitude of the **resistive fraction** may be written as

$$\frac{F_0}{\omega}\frac{r}{|Z_m|^2}=\frac{F_0\omega r}{m^2\left(\omega_0^2-\omega^2\right)^2+\omega^2 r^2}$$

Significance of the two components of the displacement curve (3)

- This is clear that the **reactive fraction** becomes **zero** and **resistive fraction is near its maximum at $\omega = \omega_0$**.
- However, they combine to give a maximum at ω, the resonant frequency for amplitude displacement, where

$$\omega^2 = \omega_r^2 = \omega_0^2 - \frac{r^2}{2m^2}$$

Problem 3.9

The equation $\ddot{x} + \omega_0^2 x = (-eE_0/m)\cos\omega t$ describes the motion of a bound undamped electric charge $-e$ of mass m under the influence of an alternating electric field $E = E_0 \cos\omega t$. For an electron number density n show that the induced polarizability per unit volume (the dynamic susceptibility) of a medium

$$\chi_e = -\frac{n\,ex}{\varepsilon_0 E} = \frac{n\,e^2}{\varepsilon_0 m(\omega_0^2 - \omega^2)}$$

(The permittivity of a medium is defined as $\varepsilon = \varepsilon_0(1 + \chi)$ where ε_0 is the permittivity of free space. The relative permittivity $\varepsilon_r = \varepsilon/\varepsilon_0$ is called the dielectric constant and is the square of the refractive index when E is the electric field of an electromagnetic wave.)

TRY Problem 3.10

Solution

- From the beginning of the unit, with the equation in the form of $m\ddot{x} + r\dot{x} + sx = F_0 \cos \omega\, t.$
- The corresponding displacement is found to be $x = \frac{F_0}{\omega|\mathrm{Z_m}|} \sin(\omega t - \phi)\,.$
- However, this case is undamped. $|Z_m| = \omega m - \frac{s}{\omega}$ and $\phi = \frac{\pi}{2}$.
- This leads to $x = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t\,.$
- Because $E = E_0 \cos \omega t$, the driving force on a bound undamped electric charge –*e* is given by $F = -eE_0 \cos \omega t$.
- This implies that $F_0 = -eE_0$. Therefore, $x = \frac{-eE_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$.
- Therefore, the susceptibility can be written as $\chi_e = \frac{-nex}{\varepsilon_0 E} = \frac{-ne}{\varepsilon_0 E} \left(\frac{-eE_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \right) = \frac{ne^2}{\varepsilon_0 m(\omega_0^2 - \omega^2)}$ #

Power supplied to an oscillator by the driving force

- To maintain the steady state, the average power supplied by the driving force just equals that being dissipated by the frictional force.

Dissipated power by frictional force

the rate of working by the frictional force

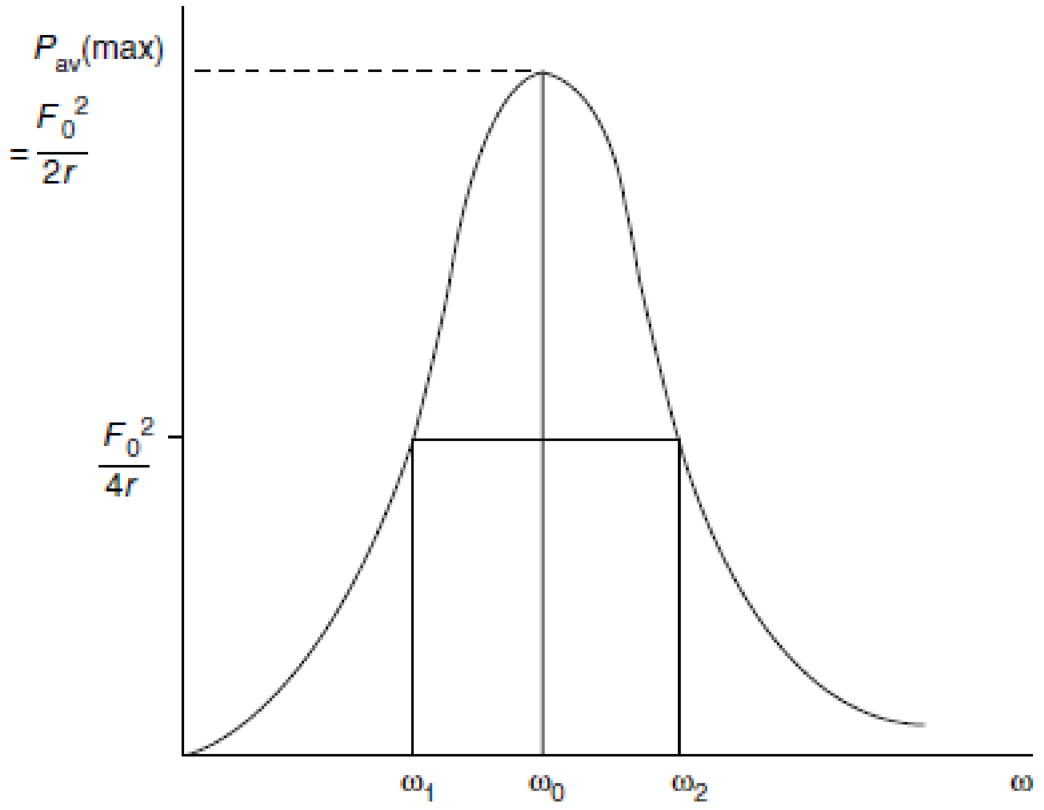
$$(r\dot{x})\dot{x} = r\left(\frac{F_0}{|Z_m|}\right)^2 \cos^2(\omega t - \phi)$$

The average of this value over one period of oscillation

$$\frac{1}{T}\int_0^T r\left(\frac{F_0}{|Z_m|}\right)^2 \cos^2(\omega t - \phi)\,dt = \frac{1}{2}\frac{rF_0^2}{|Z_m|^2} = \frac{1}{2}\frac{F_0^2}{|Z_m|}\cos\phi$$

for $\frac{r}{|Z_m|} = \cos\phi$

Variation of P_{av} with ω; Absorption resonance curve



- The maximum average power is achieved when $\cos\phi = 1$ and $Z_m = r$.
- This corresponds to the case when $\omega = \omega_0$ and velocity is in phase with applied force.
- The sharpness of the peak at resonance is determined by the value of damping constant r.
- The curve is known as the **absorption curve** of the oscillator

The Q-value in terms of the resonance absorption bandwidth

- The absorption curve in the previous slide can be used to defined the Q-value as follows

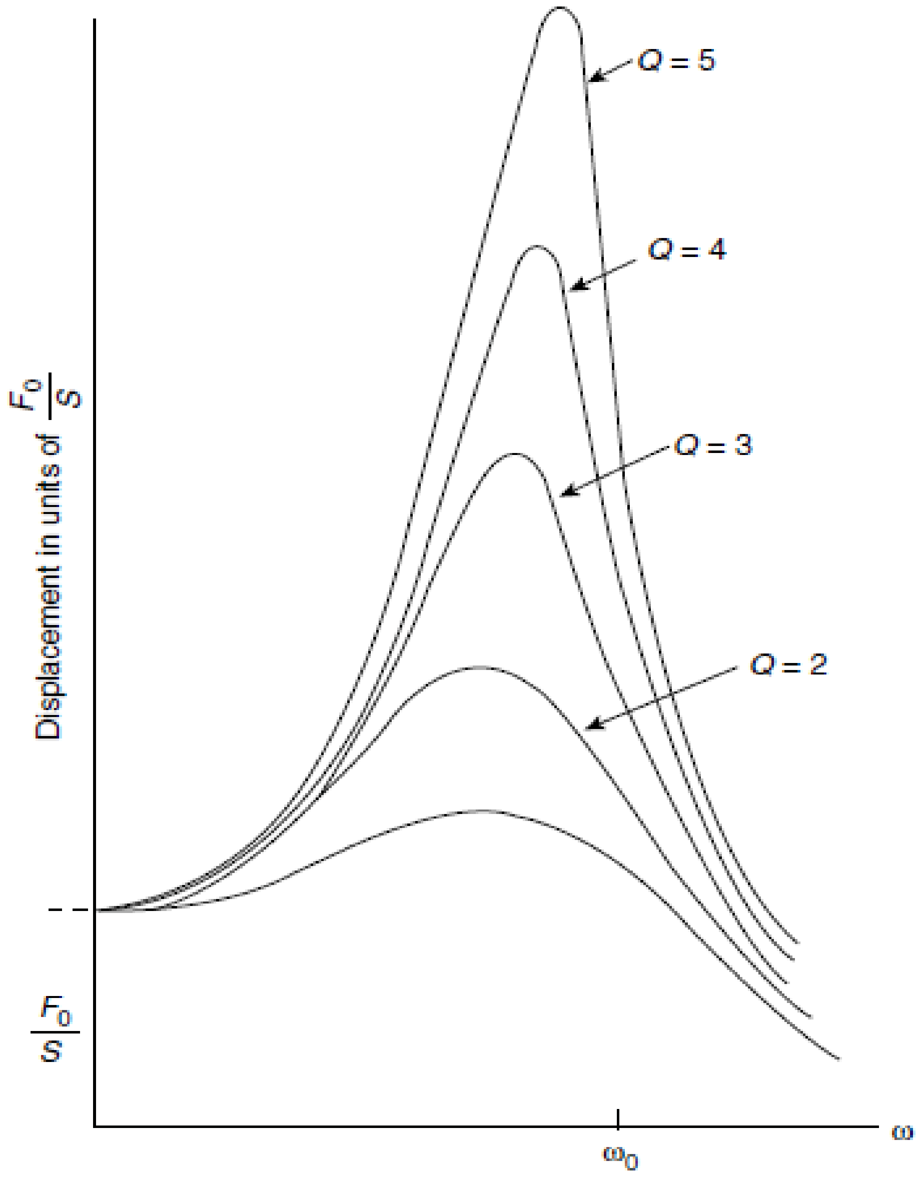
$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

- where ω_1 and ω_2 are frequencies at which the power supplied

$$P_{av} = \frac{1}{2} P_{av}(\text{maximum})$$

- And $\omega_2 - \omega_1 = \text{bandwidth}$

The Q-Value as an amplification factor



- Note that for high values of Q, the damping constant r is small.
- The displacement amplitude curve can be shown in terms of the quality factor Q of the system.

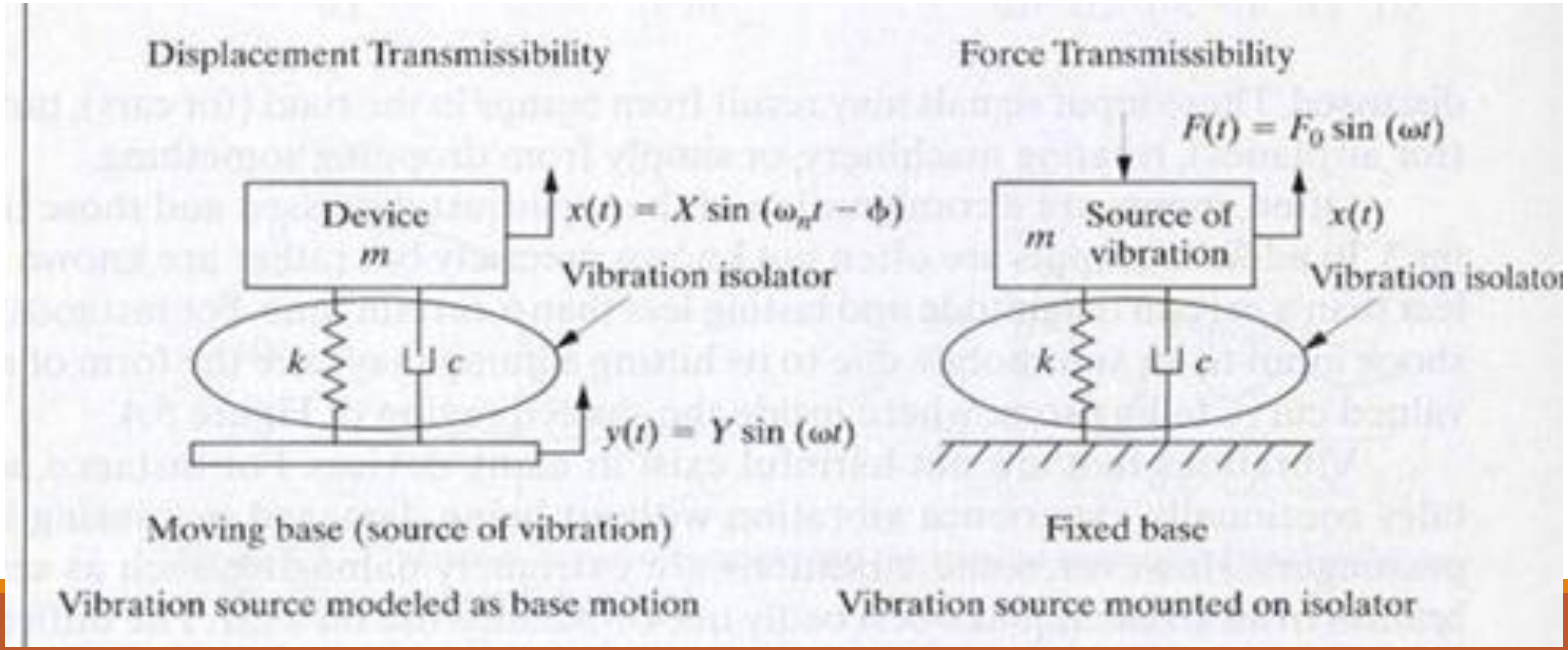
Vibration isolation

Vibration isolation

- Generally, the vibration isolation can be divided into to two basic types ; i.e.,

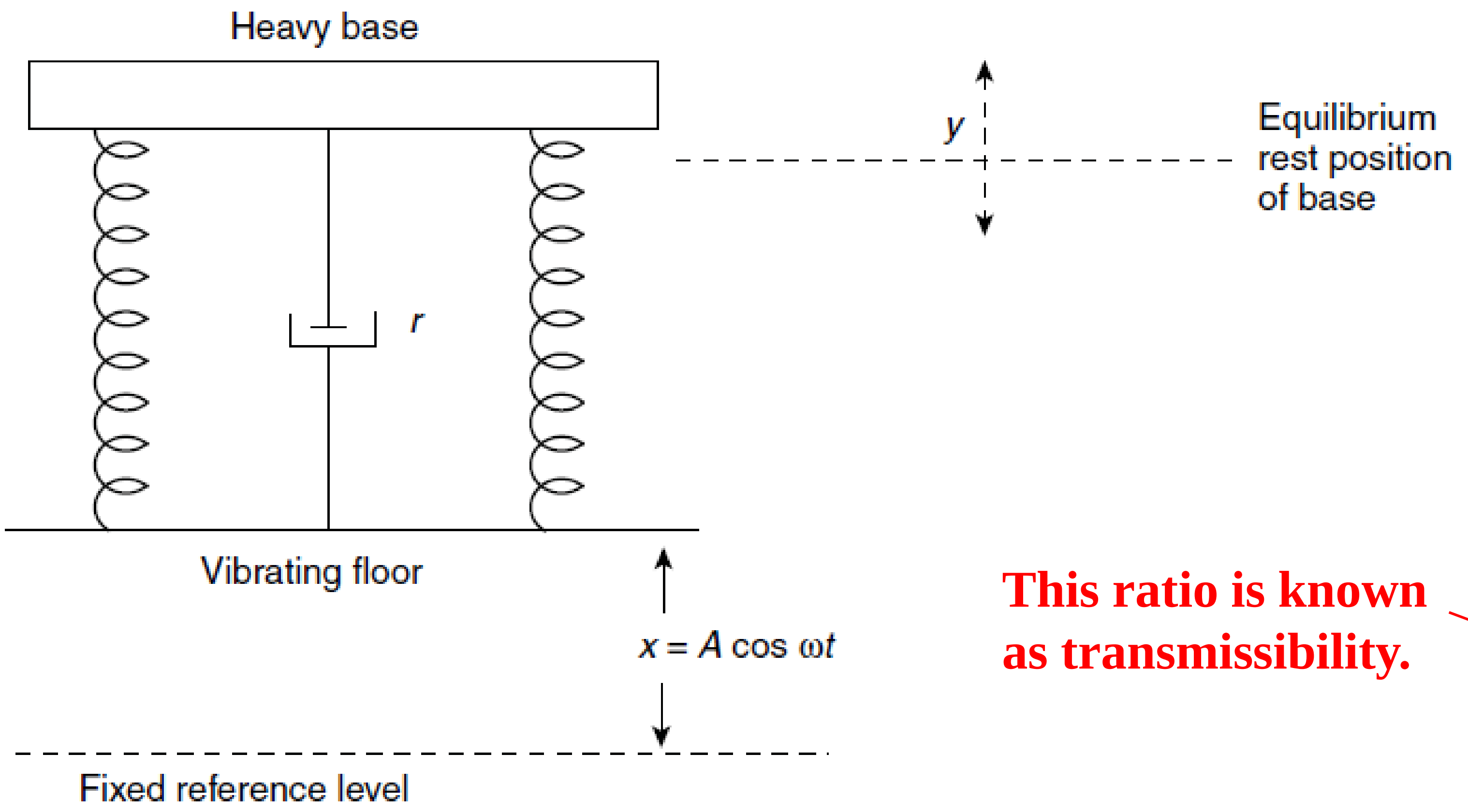
(1) displacement isolation and (2) force isolation.

- The moving-base model on **the left** is used in designing isolation to protect the device from motion of its point of attachment (base).
- The model on **the right** is used to protect the point of attachment (ground) from vibration of the mass.



http://slideplayer.com/slide/8032841/

Problem on displacement vibration insulation



This ratio is known as transmissibility.

- y = vertical displacement of the base about its rest position.
- x = vertical vibration of the floor about its equilibrium position
- **Requirement** :Protect sensitive objects (i.e. heavy base) from vibrating floors and foundations.
- **Target** : The ratio y/A is kept to a minimum.

Problem on displacement vibration insulation (cont.)

Equation of motion

- Suppose y > x; $m\ddot{y} = -s\left(y - x\right) - r\left(\dot{y} - \dot{x}\right)$

$$m\ddot{y} + r\dot{y} + sy = r\dot{x} + sx$$

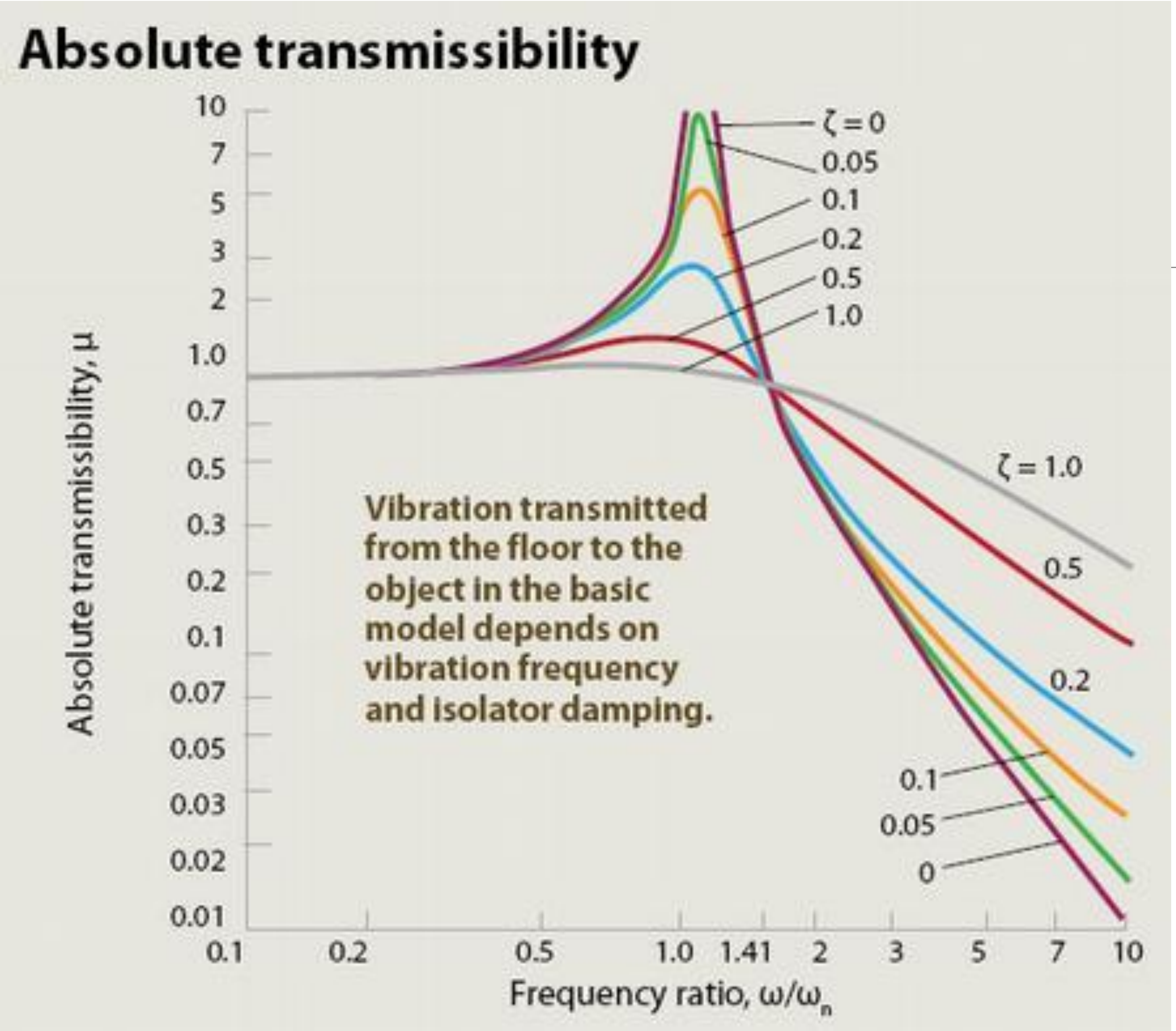
- Suppose $y = y_0 \exp\left(i\omega t\right); x = A\exp\left(i\omega t\right)$
- Determine the derivatives of *y* and *x* and substitute in the equation of motion.
- This ends up in terms of the **magnitude** ratio as follows,

$$\left|\frac{y}{A}\right| = \frac{\left(r^2 + \frac{s^2}{\omega^2}\right)^{\frac{1}{2}}}{\left(r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{\frac{1}{2}}} = \frac{\left(r^2 + \frac{s^2}{\omega^2}\right)^{\frac{1}{2}}}{|Z_m|}$$

Physical meaning of the ratio |y/A|

$$\left|\frac{y}{A}\right| = \frac{\left(r^2 + \frac{s^2}{\omega^2}\right)^{\frac{1}{2}}}{\left(r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{\frac{1}{2}}} = \frac{\left(r^2 + \frac{s^2}{\omega^2}\right)^{\frac{1}{2}}}{|Z_m|}$$

- What does it mean if the magnitude ratio is **greater than 1**?
- Under the condition, this is found that $\omega < \sqrt{2}\omega_0$ or $\frac{\omega}{\omega_0} < \sqrt{2}$



http://machinedesign.com/archive/shaking-vibration-models

Displacement Transmissibility |y/A|

- The **displacement vibration isolator** will generally operate at the **mass controlled** end of the frequency spectrum and the resonant frequency is designed to be lower than the range of frequencies likely to be met.

Analysis of the displacement transmissibility

- From the displacement transmissibility the object vibrates **less than** the supporting surface of vibrating frequency if vibration frequency $\omega \geq \sqrt{2}\omega_0$ (region of isolation).
- Lower-stiffness vibration isolators decrease the natural frequency ω_0 and transmit less vibration to the object for almost driving frequencies.
- The increasing isolator damping reduces an object's vibration amplitude at $\omega > \omega_0$ by decresing isolation at $\omega < \sqrt{2}\omega_0$ (region of amplification).

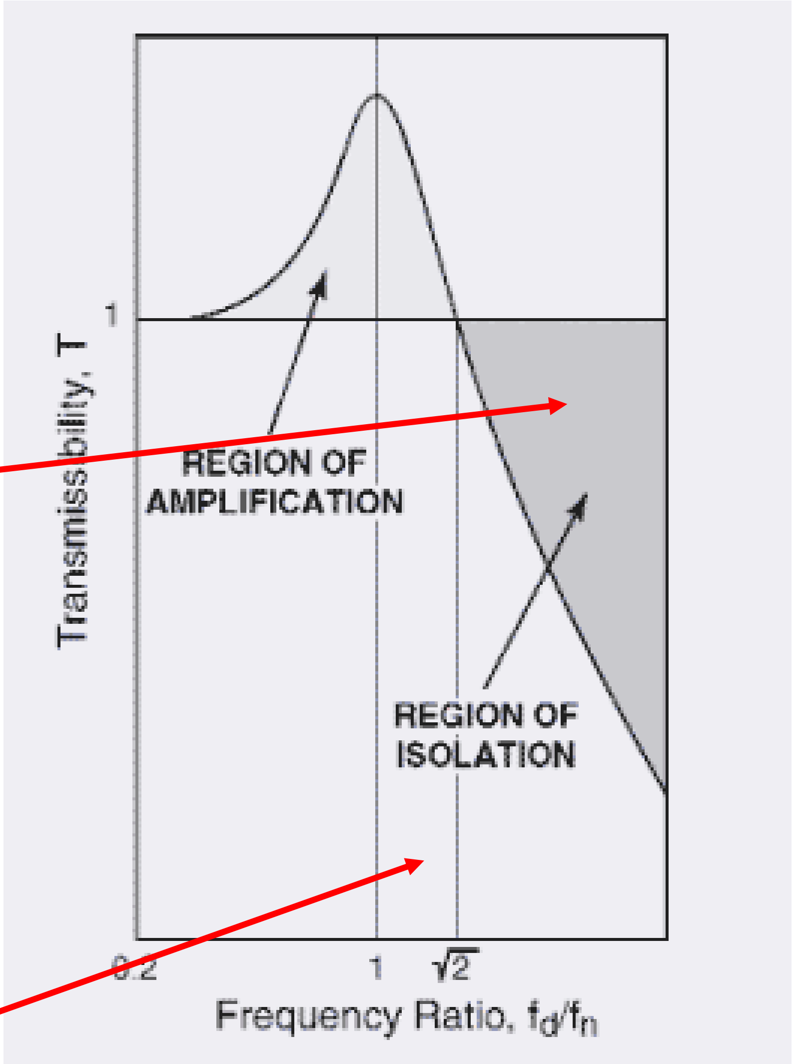


Figure 6 Typical transmissibility curve for an isolated system where fd = disturbance frequency and fn = isolation system natural frequency.

http://www.novibration.com/isoselectguide.htm

Displacement transmissibility in terms of damping ratio ζ

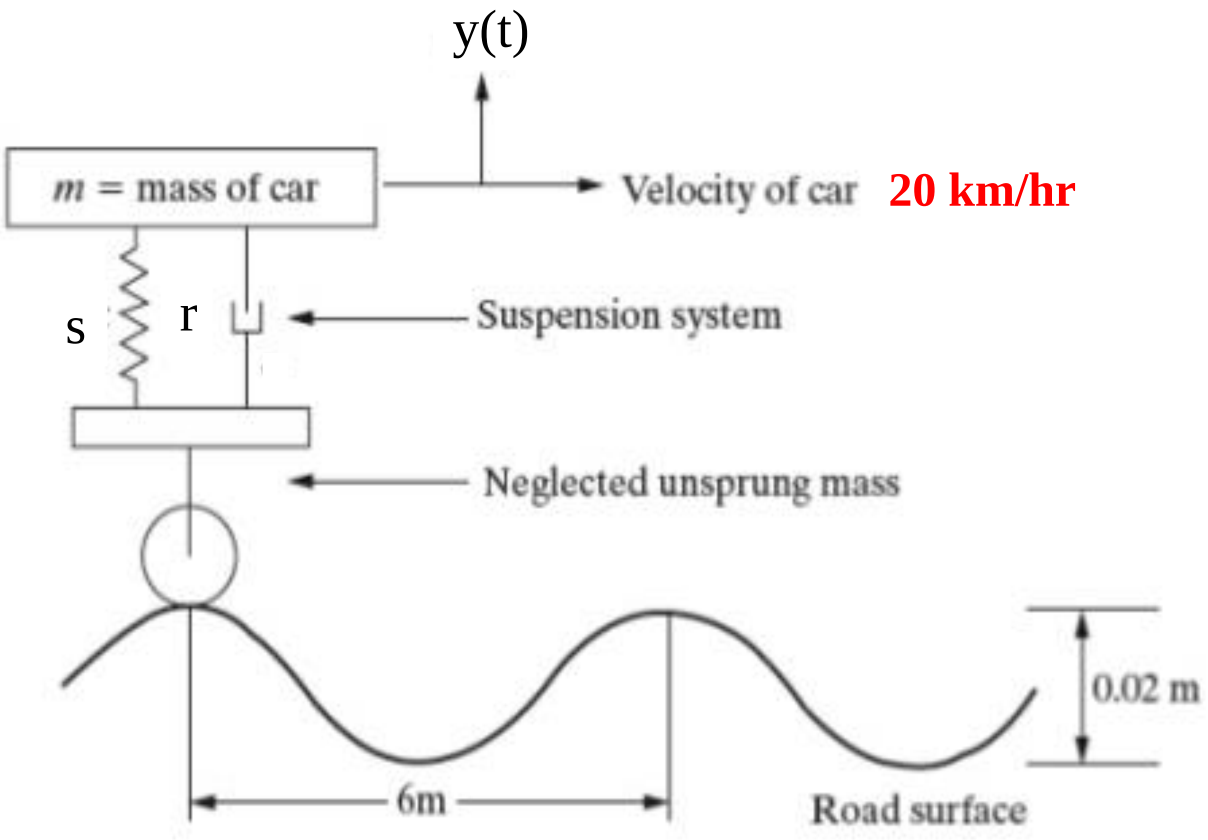
- By definition, the damping ratio ζ is given as the ratio of the damping factor to the critical damping factor,

$$\zeta = \frac{r}{r_c} = \frac{r}{2\sqrt{ms}}$$

- This leads to $\frac{r}{m} = 2\zeta\sqrt{\frac{s}{m}} = 2\zeta\omega_0$
- Therefore, the transmissibility in terms of ζ is written as

$$\text{Transmissibility} = \frac{\left[1+\left(2\zeta\frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}}{\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\zeta\frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}}$$

Problem : Effect of speed on the amplitude of car vibration



Given

(1) car speed = 20 km/hr

(2) car mass = 1007 kg

(3) stiffness s = 4 x 10^4 N/m

(4) damping constant r = 2000 Ns/m

Determine the **amplitude response** of the car to the vibrating road surface by considering the surface disturbance in the form of a **sinusoidal input**.

Model the road as a sinusoidal input to base motion of the car model

Approximation of road surface:

$$y(t) = (0.01\text{ m})\sin\omega_b t$$

$$\omega_b = v(\text{km/hr})\left(\frac{1}{0.006\text{ km}}\right)\left(\frac{\text{hour}}{3600\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{\text{cycle}}\right) = 0.2909v\text{ rad/s}$$

$$\omega_b(20\text{km/hr}) = 5.818\text{ rad/s}$$

From the data give, determine the frequency and damping ratio of the car suspension:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4\text{ N/m}}{1007\text{ kg}}} = 6.303\text{ rad/s} \quad (\approx 1\text{ Hz})$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2000\text{ Ns/m}}{2\sqrt{(4\times10^4\text{ N/m})(1007\text{ kg})}} = 0.158$$

13

$$r = \frac{\omega_b}{\omega} = \frac{5.818}{6.303}$$

$$X = Y\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

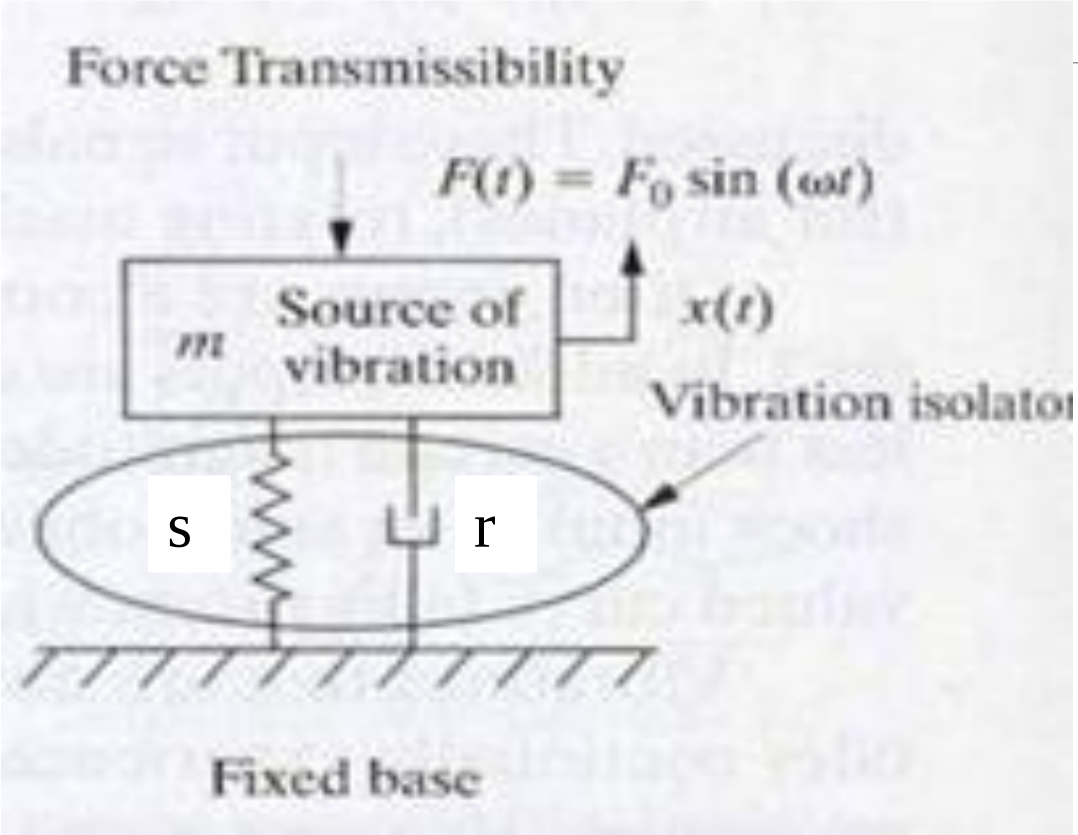
$$= (0.01\text{ m})\sqrt{\frac{1+[2(0.158)(0.923)]^2}{\left(1-(0.923)^2\right)^2+\left(2(0.158)(0.923)\right)^2}} = 0.0319\text{ m}$$

Alternatively, this formula can be used

$$\left|\frac{y}{A}\right| = \frac{\left(r^2+\frac{s^2}{\omega^2}\right)^{\frac{1}{2}}}{|Z_m|}$$

At the end, y = 0.031 m

Force vibration isolation



- The vibration source is mounted on isolator composed of a spring with stiffness s and a damper with damping constant r .
- The mass is disturbed by a force F(t).
- What is the force transmissibility for isolating the source of vibration?

http://slideplayer.com/slide/8032841/

Force vibration isolation (cont.)

- Equation of motion of mass m is given by $m\ddot{x} + r\dot{x} + sx = F_0 \sin \omega t$
- The solution as the displacement is written as

$$x = \frac{-F_0}{\omega \left| Z_m \right|} \cos(\omega t - \phi)$$

- This can be written in terms of ζ, ω and ω_0 as

$$x = \frac{-F_0}{\omega \left[\left(1 - \left(\frac{\omega}{\omega_0} \right)^2 \right)^2 + \left(2\zeta \frac{\omega}{\omega_0} \right)^2 \right]^{\frac{1}{2}}} \cos(\omega t - \phi)$$

Force vibration isolation (cont.)

- The response of the supporting base is due to the force combination of spring with stiffness s and damper with damping constant r .

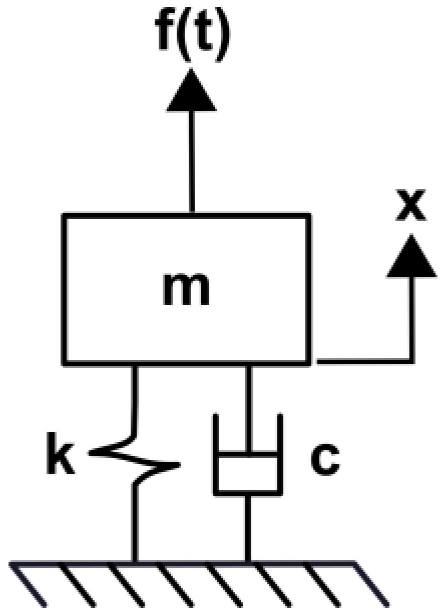
$$f(t) = sx + r\dot{x}$$

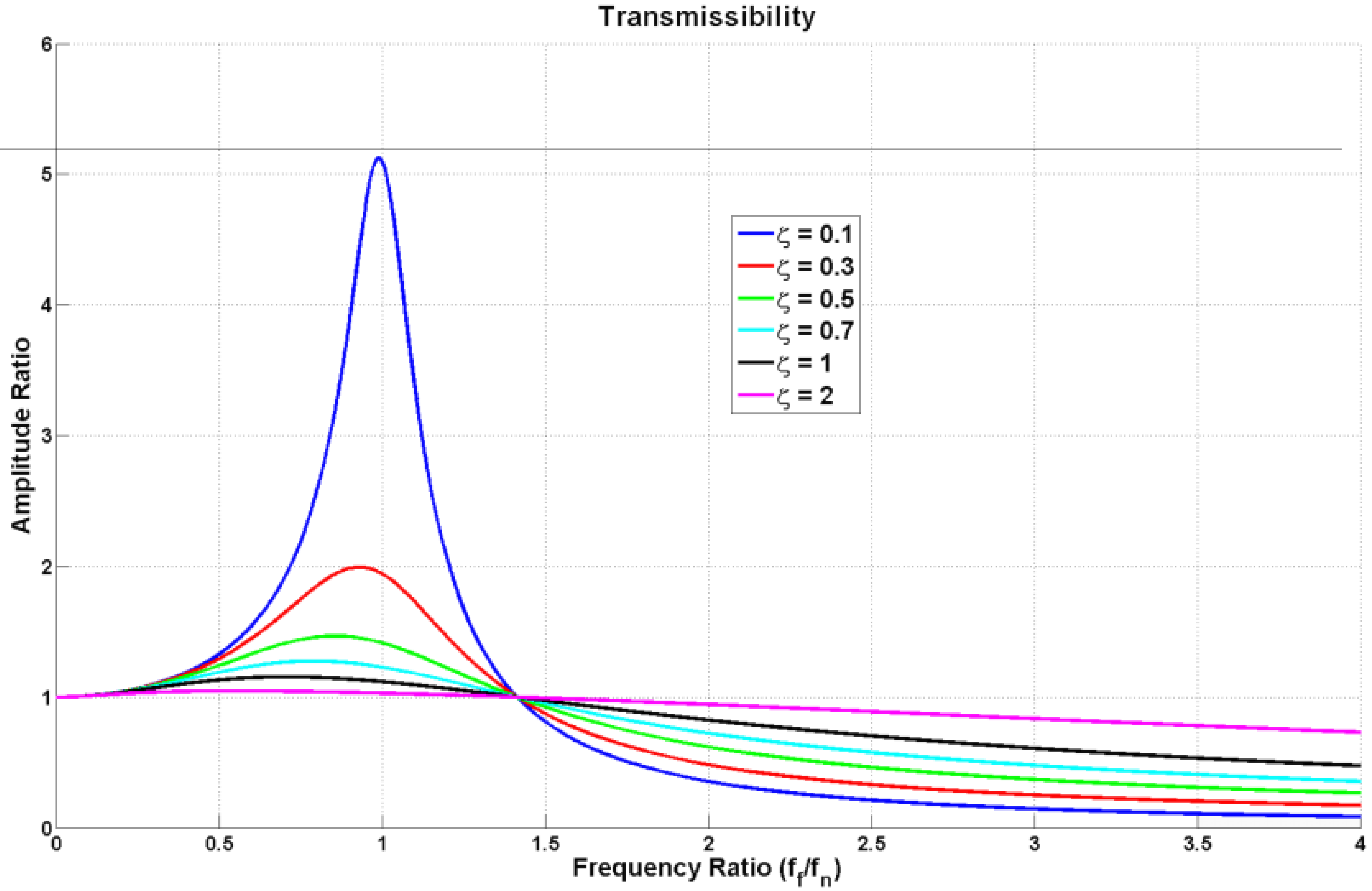
- By substituting x from the previous slide and determine the force transmissibility,

$$\frac{A}{F_0} = \frac{\left[1 + \left(2\zeta \frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}}{\left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}}$$

A = vibrating amplitude of the base

Force transmissibility curve





http://www.kaztechnologies.com/faq/

Homework #3

1. In a plasma the charges are free. Consider a free point charge q in a uniform and monochromatic electric field $\mathbf{E} = E\exp(-i\omega t)\hat{x}$, where $\hat{x}$ is the unit vector in x direction. (The physical electric field is given by the real part of $\mathbf{E}$.) Show that the displacement of the charge is

$$x = X\exp(-i\omega t), \quad X = -\frac{qE}{m\omega^2}, \tag{1}$$

where m is the mass of the charge. If there are N such free charges per unit volume, what is the polarization density associated with the charges? Argue that the relative permittivity can be written in the form

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}, \tag{2}$$

and find ω_p. Note that for $\omega < \omega_p$, ϵ_r is negative.

Homework #3 (cont.)

2. Equation of motion สำหรับ ระบบ forced mass spring damping เขียนได้เป็น

$$\ddot{x} + 2\alpha\omega\dot{x} + \omega^2 x = \omega^2 A_0 \cos\sigma t$$

โดยมี solution เขียนได้เปี่น

$$x = \frac{A_0\left[1-\left(\sigma^2/\omega^2\right)\right]\cos\sigma t + 2A_0\alpha\left(\sigma/\omega\right)\sin\sigma t}{\left(1-\left(\sigma^2/\omega^2\right)\right)^2 + 4\alpha^2\left(\sigma^2/\omega^2\right)} + A_0 e^{-\alpha\omega t}\cos\left[\left(1-\alpha^2\right)^{\frac{1}{2}}\omega t - \varepsilon\right]$$

2.1 จงวาดกราฟเฉพาะ steady state response ของระบบ forced mass spring damping นี้

2.2 จงวาดกราฟคร่าว ๆ แสดงการเปลี่ยนแปลงเฉพาะ amplitude ของ steady state response กับ σ/ω สำหรับ

ค่า $\alpha = 0, 0.2, 0.4$ และ 1.0